PROBLEM SET 9

Note Title 4/21/2009

S6 (a)

$$\mathcal{L}\{te^{-at}\} = \int_0^\infty t e^{-at} e^{-st}\, dt$$

$$= \int_0^\infty t e^{-(s+a)t}\, dt$$

Integration by parts:

Let $u = t$ $\quad dv = e^{-(s+a)t}$

$du = dt$ $\quad v = -\frac{1}{(s+a)} e^{-(s+a)t}$

$$= -t \frac{1}{(s+a)} e^{-(s+a)t} \Big|_0^\infty + \frac{1}{(s+a)} \int_0^\infty e^{-(s+a)t}\, dt$$

$$= -\frac{1}{(s+a)^2} e^{-(s+a)t} \Big|_0^\infty$$

$$= \frac{1}{(s+a)^2}$$

S6 (b)

$$\mathcal{L}\left\{\frac{d}{dt}\left(te^{-at}\right)\right\} = s\,X(s) - x(0)$$

where $x(t) = te^{-at} \Rightarrow x(0) = 0$

$X(s) = \dfrac{1}{(s+a)^2}$ from part (a)

$$= \frac{s}{(s+a)^2}$$

S6 (c)

$$\frac{d}{dt}\left(te^{-at}\right) = -ate^{-at} + e^{-at}$$

$$\mathcal{L}\left\{\frac{d}{dt}\left(te^{-at}\right)\right\} = -a\,\mathcal{L}\left\{te^{-at}\right\} + \mathcal{L}\left\{e^{-at}\right\}$$

$$= \frac{-a}{(s+a)^2} + \frac{1}{(s+a)}$$

$$= \frac{-\cancel{a} + s + \cancel{a}}{(s+a)^2}$$

check

ST (a)

$$\mathcal{L}\left\{\int_0^t e^{-ax}\,dx\right\} = \frac{X(s)}{s}$$

where $X(s) = \mathcal{L}\{e^{-at}\} = \frac{1}{(s+a)}$

$$\therefore \quad = \frac{1}{s(s+a)}$$

ST (b)

$$\mathcal{L}\left\{\int_0^t y\,dy\right\} = \frac{X(s)}{s}$$

where $X(s) = \mathcal{L}\{t\} = \frac{1}{s^2}$

$$= \frac{1}{s}\frac{1}{s^2}$$

$$= \frac{1}{s^3}$$

S6 (c)

$$\int_0^t e^{-ax}\,dx = -\frac{1}{a} e^{-ax} \Big|_0^t = \frac{1}{a} - \frac{1}{a} e^{-at}$$

$$\mathcal{L}\left\{\int_0^t e^{-ax}\,dx\right\} = \frac{1}{a}\mathcal{L}\{1\} - \frac{1}{a}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{a}\frac{1}{s} - \frac{1}{a}\frac{1}{(s+a)}$$

$$= \frac{1}{\cancel{a}}\frac{\cancel{s}+\cancel{a}-\cancel{s}}{s(s+a)}$$

$$= \frac{1}{s(s+a)}$$ check.

$$\int_0^t y\,dy = \frac{1}{2} y^2 \Big|_0^t = \frac{t^2}{2}$$

$$\mathcal{L}\left\{\frac{t^2}{2}\right\} = \frac{1}{2}\mathcal{L}\{t^2\}$$

$$= \frac{1}{2}(-1)^2 \frac{d}{ds} X(s)$$

where $X(s) = \mathcal{L}\{1\} = \frac{1}{s}$

$$\frac{d}{ds} X(s) = -\frac{1}{s^2}$$

$$\frac{d^2}{ds^2} X(s) = \frac{2}{s^3}$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}}{s^3}$$

$$= \frac{1}{s^3}$$

check.

S8 (a)

$$\mathcal{L}\left\{\frac{d}{dt}\sin\omega t\right\} = s X(s) - x(0)$$

where $X(s) = \mathcal{L}\{\sin t\} = \dfrac{\omega}{s^2+\omega^2}$

$$= \frac{s\omega}{s^2+\omega^2}$$

$x(0) = 0$

S8 (b)

$$\mathcal{L}\left\{\frac{d}{dt}\cos\omega t\right\} = s X(s) - x(0)$$

where $X(s) = \mathcal{L}\{\cos\omega t\} = \dfrac{s}{s^2+\omega^2}$

$x(0) = 1$

$$= s\,\frac{s}{s^2+\omega^2} - 1$$

$$= \frac{\cancel{s^2} - \cancel{s^2} - \omega^2}{s^2+\omega^2}$$

$$= -\frac{\omega^2}{s^2+\omega^2}$$

S8 (c)

$$\mathcal{L}\left\{\frac{d^3}{dt^3}t^2\right\} = s^3 X(s) - s^2 x(0) - s^1 x'(0) - \overset{1}{\cancel{s^0}}\, x''(0)$$

where $X(s) = \mathcal{L}\{t^2\} = \dfrac{2}{s^3}$

$x(t) = t^2$

$x'(t) = 2t$

$x''(t) = 2$

$$= \cancel{s^3}\,\frac{2}{\cancel{s^3}} - 1(2)$$

$$= 0$$

38 (d)

$$\mathcal{L}\left\{\frac{d}{dt}\sin\omega t\right\} = \mathcal{L}\left\{\omega\cos(\omega t)\right\}$$

$$= \omega\,\mathcal{L}\left\{\cos\omega t\right\}$$

$$= \frac{s\omega}{s^2+\omega^2}$$ check

$$\mathcal{L}\left\{\frac{d}{dt}\cos(\omega t)\right\} = \mathcal{L}\left\{-\omega\sin\omega t\right\}$$

$$= -\omega\,\mathcal{L}\left\{\sin\omega t\right\}$$

$$= -\omega\frac{\omega}{s^2+\omega^2}$$

$$= -\frac{\omega^2}{s^2+\omega^2}$$ check

$$\mathcal{L}\left\{\frac{d^3 t^2}{dt^3}\right\} = \mathcal{L}\left\{\cancelto{0}{\frac{d}{dt}2}\right\}$$

$$= 0$$ check

S9 (a)

$$\mathcal{L}\left\{-20e^{-5(t-2)}u(t-2)\right\} = -20\,\mathcal{L}\left\{e^{-5(t-2)}u(t-2)\right\}$$

$$= -20\,e^{-2s}\,\mathcal{L}\left\{e^{-5t}u(t)\right\}$$

$$= -20\,e^{-2s}\,\frac{1}{s+5}$$

$$= -\frac{20\,e^{-2s}}{s+5}$$

S9 (b)

$$x(t) = (8t-8)\left[u(t-1) - u(t-2)\right]$$
$$+ (24-8t)\left[u(t-2) - u(t-4)\right]$$
$$+ (8t-40)\left[u(t-4) - u(t-5)\right]$$

$$= (8t-8)\,u(t-1)$$
$$+ \left[(24-8t) - (8t-8)\right]u(t-2)$$

$$+ \left[(8t-40) - (24-8t)\right] u(t-4)$$
$$- (8t-40) \quad u(t-5)$$

$$= 8(t-1) \, u(t-1)$$
$$- 16(t-2) \, u(t-2)$$
$$+ 16(t-4) \, u(t-4)$$
$$- 8(t-5) \, u(t-5)$$

$$\mathcal{L}\{x(t)\} = 8 \, \mathcal{L}\{(t-1) \, u(t-1)\}$$
$$- 16 \, \mathcal{L}\{(t-2) \, u(t-2)\}$$
$$+ 16 \, \mathcal{L}\{(t-4) \, u(t-4)\}$$
$$- 8 \, \mathcal{L}\{(t-5) \, u(t-5)\}$$

$$= 8 \left[\frac{e^{-s}}{s^2} - 2 \frac{e^{-2s}}{s^2} + 2 \frac{e^{-4s}}{s^2} - \frac{e^{-5s}}{s^2} \right]$$

$$= 8 \frac{\left[e^{-s} - 2e^{-2s} + 2e^{-4s} - e^{-5s} \right]}{s^2}$$